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On the interaction between internal gravity waves and a shear flow

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The theory of wave action conservation is summarized, and its interpretation in terms of the working, against the rate of strain of the basic flow, of an interaction stress associated with the waves is discussed. Usually this interaction stress is identical with the radiation stress of a uniform plane wave. The problem of internal gravity wave propagation in an incompressible, stratified Boussinesq liquid is considered in detail for a more general basic flow than has hitherto been treated, and the interaction stress is derived. One component of the interaction stress tensor is only equal to the corresponding component of the radiation stress tensor if we include in the latter, in addition to the Reynolds stress, a term associated with the redistribution of matter, on the average, by the wave. Two other components of the radiation stress tensor are modified in a similar manner, but the corresponding components of the interaction stress tensor are undefined, and so no comparison is possible.

1. Introduction

Considerable attention has been given to the problem of the propagation of non-dissipative dispersive hydrodynamic waves of small amplitude in inhomogeneous media in non-uniform mean motion. A simplifying assumption of great use and wide validity is that the time and length scales of the basic flow are much greater than the period and wavelength of the waves. Particular problems investigated with this assumption are those of the propagation of surface water waves on a varying current (Longuet-Higgins & Stewart 1961, 1964; Whitham 1962) and sound waves in a non-uniform moving medium (Blokhintsev 1946). Bretherton (1966) and Hines & Reddy (1967) consider the propagation of internal gravity waves in a simple shear flow and discuss the meteorological importance of this. It is with an extension of this problem and interpretation of the results that this paper is mainly concerned.

The way in which the wave-number and frequency of a wave-train change along well-defined paths is well known, and is discussed, for example, by Bretherton &

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Garrett (1968). The frequency ω and wave-number $\mathbf{k}$ are derived, as functions of position $\mathbf{x}$ and time t, from a phase function $\theta(\mathbf{x}, t)$ by

$$\omega = -\partial\theta/\partial t, \quad \mathbf{k} = \nabla\theta. \tag{1.1}$$

ω, $\mathbf{k}$ are connected by the dispersion relation

$$\omega = \Omega(\mathbf{k}, \lambda), \tag{1.2}$$

where $\lambda(\mathbf{x}, t)$ is a parameter involving the local properties of the medium. Then changes in ω, $\mathbf{k}$ are governed by

$$\frac{d\omega}{dt} = \frac{\partial\Omega}{\partial\lambda}\frac{\partial\lambda}{\partial t}, \quad \frac{dk_i}{dt} = -\frac{\partial\Omega}{\partial\lambda}\frac{\partial\lambda}{\partial x_i}, \tag{1.3}$$

where
$$\frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{c}.\nabla$$

and $\mathbf{c}$ is the group velocity, defined by

$$c_i = \partial\Omega/\partial k_i. \tag{1.4}$$

However, until recently there seemed to be no general result governing changes in the amplitude of the waves. In media basically at rest, it was generally known that the amplitude might be derived from conservation of the energy of a wave packet. Thus, if E denotes the energy density, related to the square of the amplitude, then changes in amplitude are governed by the equation

$$dE/dt + E\nabla.\mathbf{c} = 0. \tag{1.5}$$

In the presence of a non-uniform basic flow it was recognized in many cases that the energy of a wave packet is affected by an interaction with the mean flow, and no general result was known. However, the problem is of such a clear-cut and general nature that the existence of a simple general answer might have been suspected. Such an answer has now been found. Based on work by Whitham (1965), it was suggested by Garrett (1967) and proved by Bretherton & Garrett (1968) and Bretherton (1968) that, for a wave packet in a moving and/or time-dependent medium, wave action (defined as wave energy divided by the wave frequency relative to the basic flow) is conserved. Thus the energy equation becomes

$$\frac{d}{dt}\left(\frac{E}{\omega'}\right) + \frac{E}{\omega'}\nabla.\mathbf{c} = 0, \tag{1.6}$$

where $\omega' = \omega - \mathbf{U}.\mathbf{k}$ is the frequency of the waves measured in a frame of reference moving with the local mean velocity $\mathbf{U}$ of the medium. The wave-energy density E is also measured in this frame of reference. Equation (1.5) is now a special case of (1.6) for wave propagation in a stationary time-independent medium, for which $\omega = \omega'$ and $d\omega/dt = 0$.

If the dispersion relation of the waves relative to the medium is

$$\omega' = \Omega'(\mathbf{k}, \lambda'(\mathbf{x}, t)), \tag{1.7}$$

then $\omega = \mathbf{U}.\mathbf{k} + \omega'$ has $\mathbf{U}$ and λ' as components of the parameter λ occurring in (1.2). Thus

$$\frac{d\omega}{dt} = k_j\frac{\partial U_j}{\partial t} + \frac{\partial\Omega'}{\partial\lambda'}\frac{\partial\lambda'}{\partial t} \tag{1.8}$$

and
$$\frac{dk_i}{dt} = -k_j \frac{\partial U_j}{\partial x_i} - \frac{\partial \Omega'}{\partial \lambda'} \frac{\partial \lambda'}{\partial x_i}. \tag{1.9}$$

Hence
$$\frac{d\omega'}{dt} = -k_j c_i' \frac{\partial U_j}{\partial x_i} + \frac{\partial \Omega'}{\partial \lambda'} \frac{D\lambda'}{Dt}, \tag{1.10}$$

where
$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{U}.\nabla$$

and
$$c_i' = c_i - U_i = \partial \Omega' / \partial k_i.$$

Changes in ω' given by (1.10) mean that a wave packet is exchanging energy with the mean flow. The way in which this energy exchange may be interpreted in terms of the rate of working, against the rate of strain of the basic flow, of an 'interaction stress' associated with the waves, is discussed in §3. Usually the identification of this interaction stress is straightforward. For example, for surface gravity waves it may be shown (Bretherton & Garrett 1968) that the conservation of wave action is equivalent to the energy equation of Longuet-Higgins & Stewart (1964). Thus our interaction stress is the same as their radiation stress, which is just the second-order mean momentum flux for a uniform plane wave, and is composed of the Reynolds stresses plus the second-order mean pressure.

However, consideration of the propagation of internal gravity waves in a more general basic flow than was taken by Bretherton (1966) or Hines & Reddy (1967) leads to a slightly unexpected term in the interaction stress. This, and the way in which it may be identified with a contribution to the radiation stress, will be discussed later, but, before turning to the energy exchange in general, and for internal gravity waves in particular, it seems worthwhile to give a summary of the derivation of wave action conservation.

2. Wave action conservation

Whitham (1965) assumed that the Eulerian equations of motion governing a system which can support non-linear plane waves can be derived from a variational principle
$$\delta \int L \, dV \, dt = 0, \tag{2.1}$$
where L is a function of certain potentials describing the problem and their derivatives, dV is an element of volume and dt an element of time. Whitham then considered a slowly varying wave-train and assumed that the behaviour of this could be derived from an 'averaged variational principle'
$$\delta \int \mathscr{L} \, dV \, dt = 0, \tag{2.2}$$
where $\mathscr{L}$ is the time average of L over a local period.

Whitham's emphasis was on non-linear waves, but he also considered the implications of his theory for the linearized case, i.e. for small amplitude waves on a stationary basic state.

In this case
$$\mathscr{L} = \mathscr{L}(a, \omega, \mathbf{k}; \lambda), \tag{2.3}$$
where a is the amplitude, ω the frequency and $\mathbf{k}$ the wave-number. As before λ is a parameter involving the local properties of the medium. $\mathscr{L}$ is proportional to a^2 and given by
$$\mathscr{L} = G(\omega, \mathbf{k}; \lambda) a^2. \tag{2.4}$$

The averaged variational principle then gives

$$G = 0 \tag{2.5}$$

from variation of a, and so $\mathscr{L} = 0$. Equation (2.5) must be the local dispersion relation, equivalent to (1.2) here. From variation of the phase function θ, Whitham obtained

$$\frac{\partial \mathscr{L}_\omega}{\partial t} - \frac{\partial \mathscr{L}_{ki}}{\partial x_i} = 0. \tag{2.6}$$

But the group velocity $\mathbf{c}$ is given by

$$c_i = \left.\frac{\partial \omega}{\partial k_i}\right|_{\mathscr{L}=0} = -\frac{\mathscr{L}_{ki}}{\mathscr{L}_\omega}, \tag{2.7}$$

and so (2.6) becomes

$$d\mathscr{L}_\omega/dt + \mathscr{L}_\omega \nabla . \mathbf{c} = 0. \tag{2.8}$$

Whitham also pointed out that, as a consequence of Noether's theorem in the calculus of variations, then

$$\frac{\partial}{\partial t}(\omega \mathscr{L}_\omega - \mathscr{L}) - \frac{\partial}{\partial x_i}(\omega \mathscr{L}_{ki}) = 0 \tag{2.9}$$

if $\mathscr{L}$ has no explicit dependence on t. $\omega\mathscr{L}_\omega - \mathscr{L}$ is thus identified with the energy density E, and $-\omega\mathscr{L}_{ki}$ with the energy flux. Thus, as $\mathscr{L} = 0$,

$$\mathscr{L}_\omega = E/\omega, \tag{2.10}$$

though this was not explicitly stated by Whitham, and (2.8) gives the conservation of wave action for small amplitude waves propagating in a non-uniform and time-dependent, but stationary, medium. Garrett (1967) pointed out that a way to change the 'intrinsic' frequency of a wave packet, other than by having the medium stationary but time dependent, is to allow the waves to propagate in a medium in non-uniform mean motion. Hence one is led to suspect that conservation of wave action, as described by (1.6), might be a general result.

In certain particular examples Whitham stated what his Lagrangian L is, and justified the use of the averaged variational principle, but, for the rigorous justification of wave action conservation for the time-dependent and moving-media problems, several extensions and modifications needed to be made. These have been carried out by Bretherton & Garrett (1968) and Bretherton (1968) in the manner outlined below.

On the basis of work by Eckart (1963), a form of Hamilton's principle is obtained for small perturbations about a state of motion which is itself a solution of the equations of motion. Eckart's principle is based on that due to Herivel (1955) and is valid for any inviscid, adiabatic, compressible, unbounded fluid in a gravitational field. This is extended to include conservative lateral boundary conditions (as is necessary for surface water waves, for example), and to allow for incompressibility. If the fluid is perfectly conducting, then the effects of a frozen-in magnetic field may be included, as in the variational principle of Lundgren (1963). The important difference between this and Whitham's approach is that by describing the motion in terms of the particles (a Lagrangian description) instead of positions in space (an Eulerian description) it is possible to say exactly what the variational principle is, namely Hamilton's principle, and the Lagrangian, L, can easily be written down.

Allowing for the possibility of lateral co-ordinates, Bretherton (1968) derives the averaged variational principle, where $\mathscr{L}$ is obtained from L by integration over the lateral co-ordinates as well as averaging over a period. The conservation of $\mathscr{L}_\omega$ in the sense of (2.8) then follows as before, but this still has to be identified with E/ω'.

If there exists a frame of reference with respect to which the medium is locally at rest, then

$$E = \omega' \mathscr{L}_{\omega'} - \mathscr{L} \tag{2.11}$$

as $\omega' \mathscr{L}_{\omega'} - \mathscr{L}$ is found to give the perturbation energy density for waves on a stationary basic state (this is a consequence of the basic variational principle being Hamilton's principle). But $\mathscr{L} = 0$ and

$$\mathscr{L}(a, \omega, \mathbf{k}; \lambda) = \mathscr{L}(a, \omega' + \mathbf{U}.\mathbf{k}, \mathbf{k}; \lambda). \tag{2.12}$$

Thus
$$\mathscr{L}_\omega = \mathscr{L}_{\omega'} = E/\omega',$$

finally giving (1.6) from (2.8).

3. The energy exchange

If there were no interaction between the waves and the basic state of the medium, then the energy of a group of waves would be conserved and the energy equation would be equation (1.5). Now (1.6) may be written

$$\frac{dE}{dt} + E\nabla.\mathbf{c} - \frac{E}{\omega'}\frac{d\omega'}{dt} = 0 \tag{3.1}$$

and so $-(E/\omega')d\omega'/dt$ in some way represents the interaction between the waves and the basic flow. But $d\omega'/dt$ is given by (1.10), and so (3.1) becomes

$$\frac{dE}{dt} + E\nabla.\mathbf{c} + \frac{Ek_j c_i'}{\omega'}\frac{\partial U_j}{\partial x_i} - \frac{E}{\omega'}\frac{\partial \Omega'}{\partial \lambda'}\frac{D\lambda'}{Dt} = 0. \tag{3.2}$$

For every type of wave motion that has been studied in detail, it turns out that, for a basic state satisfying the requirements of slow variation mentioned in §1, $D\lambda'/Dt$ satisfies an equation of the form

$$\frac{1}{\lambda'}\frac{D\lambda'}{Dt} + \Lambda_{ij}\frac{\partial U_j}{\partial x_i} = 0, \tag{3.3}$$

where Λ_{ij} is a tensor independent of $\mathbf{U}$ and its derivatives. Thus

$$\frac{dE}{dt} + E\nabla.\mathbf{c} + T_{ij}\frac{\partial U_j}{\partial x_i} = 0 \tag{3.4}$$

and the rate of loss of wave energy within a volume, each point of which moves with the group velocity, may be equated to the rate of working, against the rate of strain of the basic flow, of an interaction stress tensor T_{ij}, given by

$$T_{ij} = \frac{E}{\omega'}\left(\lambda' \frac{\partial \Omega'}{\partial \lambda'}\Lambda_{ij} + c_i' k_j\right). \tag{3.5}$$

For particular types of wave propagation, the components of T_{ij} occurring in the energy equation for waves in a basic flow satisfying the requirements of slow variation may be identified with the corresponding components of S_{ij}, the radiation stress tensor for the waves. This is defined as the second-order mean of the flux in the i direction of j momentum for a uniform plane wave.

Section 4 will be devoted to a derivation, from the equations of motion, of an energy equation for internal gravity waves in a fairly general basic flow. This will be shown to be equivalent to (1.6), and the version of (3.3) for this problem will be derived. However, it will be seen that the interaction tensor T_{ij} contains a term additional to those at first expected in S_{ij}. In §5 this extra term will be explained in terms of a somewhat unsuspected contribution to the radiation stress.

4. Internal gravity waves in a shear flow

We shall now consider the propagation of internal gravity waves in an incompressible stratified fluid with a slowly varying basic state specified by a density field $\rho(z,t)$, velocity field $\mathbf{U} = (U(x,y,z,t), V(x,y,z,t), W(z,t))$ and pressure $p(x,y,z,t)$. x, y are horizontal co-ordinates and z is measured vertically upwards. It seems that if we were to allow W to depend on x, y as well as z, then requirements of slow variation would require $\partial W/\partial x$ and $\partial W/\partial y$ to be $O(\epsilon^2)$ compared with $\partial W/\partial z$, and so negligible to our approximation, where ϵ is a small parameter expressing the ratio of a wavelength or period to the length or time scales of the basic flow. The basic velocity field as given above thus seems to be the most general which we may usefully take; it is rather artificial but will help us to understand the energy exchange. As pointed out by Bretherton (1966), who considered internal gravity wave propagation in a shear flow $(U(z), V(z), 0)$, the Richardson number $N^2/(U_z^2+V_z^2)$ must be large if the requirements of slow variation are to be satisfied.

The governing equations for the basic state are

$$\rho\, D\mathbf{U}/Dt - \rho\mathbf{g} + \nabla p = 0, \tag{4.1}$$

$$\nabla . \mathbf{U} = 0, \tag{4.2}$$

$$D\rho/Dt = 0, \tag{4.3}$$

where $\mathbf{g} = (0,0,-g)$. We now consider perturbations $\delta, \pi, \mathbf{u} = (u,v,w)$ of the density, pressure and velocity fields. δ gives rise to a perturbation buoyancy force $b = g\delta/\rho$, but making the Boussinesq approximation we neglect other terms involving δ. The linearized perturbation equations are then

$$\rho\left(\frac{Du}{Dt} + u\frac{\partial U}{\partial x} + v\frac{\partial U}{\partial y} + w\frac{\partial U}{\partial z}\right) + \pi_x = 0, \tag{4.4}$$

$$\rho\left(\frac{Dv}{Dt} + u\frac{\partial V}{\partial x} + v\frac{\partial V}{\partial y} + w\frac{\partial V}{\partial z}\right) + \pi_y = 0, \tag{4.5}$$

$$\rho\left(\frac{Dw}{Dt} + w\frac{\partial W}{\partial z}\right) + \rho b + \pi_z = 0, \tag{4.6}$$

$$\nabla . \mathbf{u} = 0, \tag{4.7}$$

$$Db/Dt = N^2 w, \tag{4.8}$$

where $D/Dt = (\partial/\partial t) + \mathbf{U}\,.\nabla$ as before, and N is the local Brunt–Väisälä frequency given by

$$N^2 = -\frac{g}{\rho}\frac{\partial\rho}{\partial z}. \tag{4.9}$$

The solution of these equations may be investigated by the standard W.K.B. method, involving an asymptotic expansion of the full solution in powers of ϵ. The first approximation, giving the locally uniform plane wave of relative frequency ω' and wave-number $\mathbf{k} = (k, l, m)$, is given by the solution of equations (4.4) to (4.8), ignoring the terms involving derivatives of $\mathbf{U}$ and treating $\mathbf{U}$, ρ as constant. Then, if

$$\mathbf{u}, b, \pi = \mathrm{Re}\,(\mathbf{u}_0, b_0, \pi_0 \exp[i(\mathbf{k}\,.\,\mathbf{x} - \omega t)]), \tag{4.10}$$

we have

$$\omega - \mathbf{U}\,.\,\mathbf{k} = \omega' = \pm\frac{N(k^2+l^2)^{\frac{1}{2}}}{(k^2+l^2+m^2)^{\frac{1}{2}}} \tag{4.11}$$

and

$$\left.\begin{aligned} u_0 &= k\pi_0/\rho\omega', \\ v_0 &= l\pi_0/\rho\omega', \\ w_0 &= -\frac{(k^2+l^2)\pi_0}{\rho m\omega'}, \\ b_0 &= -\frac{i(k^2+l^2)N^2\pi_0}{\rho m\omega'^2}. \end{aligned}\right\} \tag{4.12}$$

The local energy density E is given by

$$E = \tfrac{1}{2}\rho\overline{\mathbf{u}^2} + \tfrac{1}{2}\rho(\overline{b^2}/N^2), \tag{4.13}$$

where the bar denotes the average over a local period. Then

$$E = \tfrac{1}{4}\rho|\mathbf{u}_0|^2 + \tfrac{1}{4}\rho(|b_0|^2/N^2). \tag{4.14}$$

The following relations, which we shall use shortly, are easily verified from the above solution for a plane wave:

$$\left.\begin{aligned} \overline{\pi\mathbf{u}} &= \mathbf{c}'E, \\ \rho\overline{u^2} &= \frac{Ekc_1'}{\omega'}, \\ \rho\overline{v^2} &= \frac{Elc_2'}{\omega'}, \\ \rho\overline{w^2} &= \frac{Emc_3'}{\omega'} + E, \\ \rho\overline{uv} &= \frac{Ekc_2'}{\omega'} = \frac{Elc_1'}{\omega'}, \\ \rho\overline{uw} &= \frac{Ekc_3'}{\omega'} = \frac{Emc_1'}{\omega'} - \frac{Ekm}{k^2+l^2}, \\ \rho\overline{vw} &= \frac{Elc_3'}{\omega'} = \frac{Emc_2'}{\omega'} - \frac{Elm}{k^2+l^2}. \end{aligned}\right\} \tag{4.15}$$

As a group of waves, described locally by the plane wave solution above, moves through the basic state with the group velocity, changes in ω, $\mathbf{k}$ are governed by (1.8) and (1.9), where $\lambda' = N$. Changes in amplitude will be governed by (1.6), but we shall first check this by deriving an energy equation from (4.4) to (4.8). Taking $u \times (4.4) + v \times (4.5) + w \times (4.6)$ and using (4.3) and (4.7), we have

$$\frac{D}{Dt}(\tfrac{1}{2}\rho\mathbf{u}^2) + \rho u_i u_j \frac{\partial U_j}{\partial x_i} + \nabla.(\pi\mathbf{u}) + \rho w b = 0. \tag{4.16}$$

(In the second term, of course, $\partial W/\partial x = \partial W/\partial y = 0$.) From (4.8),

$$\rho w b = \frac{\rho b}{N^2}\frac{Db}{Dt} \tag{4.17}$$

$$= \frac{D}{Dt}\left(\tfrac{1}{2}\rho\frac{b^2}{N^2}\right) + \frac{\rho b^2}{N^2}\frac{1}{N}\frac{DN}{Dt}. \tag{4.18}$$

Thus (4.16) becomes

$$\frac{D}{Dt}\left(\tfrac{1}{2}\rho\mathbf{u}^2 + \tfrac{1}{2}\rho\frac{b^2}{N^2}\right) + \nabla.(\pi\mathbf{u}) + \rho u_i u_j \frac{\partial U_j}{\partial x_i} + \frac{\rho b^2}{N^2}\frac{1}{N}\frac{DN}{Dt} = 0. \tag{4.19}$$

We now obtain the energy equation by substituting in (4.19) the values of $\mathbf{u}$, b, π appropriate to a local uniform plane wave, and averaging over a period. Then, using (4.15), we obtain

$$\frac{dE}{dt} + E\nabla.\mathbf{c} + \frac{Ek_j c_i'}{\omega'}\frac{\partial U_j}{\partial x_i} + E\frac{\partial W}{\partial z} + \frac{E}{N}\frac{DN}{Dt} = 0. \tag{4.20}$$

This is not quite the form required by (3.2) (where $\lambda' = N$ and $\partial\Omega'/\partial\lambda' = \omega'/N$) for conservation of wave action, but it is readily shown to be equivalent on evaluation of $N^{-1}DN/Dt$. For (4.3) is

$$\partial\rho/\partial t + W\,\partial\rho/\partial z = 0, \tag{4.21}$$

and on differentiation with respect to z we obtain

$$\frac{\partial}{\partial t}\left(\frac{1}{\rho}\frac{\partial\rho}{\partial z}\right) + W\frac{\partial}{\partial z}\left(\frac{1}{\rho}\frac{\partial\rho}{\partial z}\right) = -\frac{1}{\rho}\frac{\partial\rho}{\partial z}\frac{\partial W}{\partial z} \tag{4.22}$$

and so

$$N^{-1}DN/Dt = -\tfrac{1}{2}\partial W/\partial z. \tag{4.23}$$

Thus the last two terms of (4.20) may be written as $-(E/N)\,DN/Dt$ and so (4.20) corresponds to (3.2) and gives conservation of wave action.

We also note that (4.23) is of the form of (3.3), where

$$\Lambda_{ij} = \tfrac{1}{2}\delta_{i3}\delta_{j3}. \tag{4.24}$$

5. Radiation stress

The results of §4 show that the interaction tensor T_{ij}, defined in §3, may be written as

$$T_{ij} = \rho\overline{u_i u_j} - \tfrac{1}{2}E\,\delta_{i3}\delta_{j3} \tag{5.1}$$

for all components of T_{ij} except T_{13} and T_{23}, which are undefined as

$$\partial W/\partial x = \partial W/\partial y = 0.$$

If we wish to identify T_{ij} with the radiation stress tensor S_{ij} we must explain a contribution $-\frac{1}{2}E$ to S_{33}, additional to the Reynolds stress $\rho\overline{w^2}$.

Considering a locally uniform plane wave, a surface of constant density has a sinusoidal perturbation on it due to the presence of the wave. Because of the basic stratification, that part of the fluid displaced upwards to above the mean height is heavier than that displaced downwards to below the mean height. There is, then, on the average, an excess of mass m, per unit horizontal area, above the mean position z of a surface of constant density. We may calculate m as follows: if at any time (ξ, η, ζ) is the displacement associated with the point (x, y, z) then we may say with sufficient accuracy that the particle at a point $(x, y, z+\zeta')$ has actually originated from the point $(x-\xi, y-\eta, z+\zeta'-\zeta)$ and so has density corresponding to this.

Thus
$$m = \overline{\int_0^{\zeta} \rho(z+\zeta'-\zeta)\,d\zeta'} \tag{5.2}$$

$$= -\tfrac{1}{2}\overline{\zeta^2}\frac{\partial \rho}{\partial z}, \tag{5.3}$$

and it is easily verified from the plane wave solution discussed in §4 that this gives
$$m = \tfrac{1}{2}E/g. \tag{5.4}$$

So, due to the gravitational field, there is a mean downward force exerted on the volume above z, greater by $\frac{1}{2}E$ than it would be without waves. Now the radiation stress S_{ij} may be regarded as the j component of the stress exerted, across a plane P perpendicular to the i axis, on the material on the positive side of P, due to the presence of the waves. Thus S_{33} contains a term $-gm$ in addition to the Reynolds stress $\rho\overline{w^2}$, i.e.

$$S_{33} = \rho\overline{w^2} - \tfrac{1}{2}E = T_{33}, \tag{5.5}$$

which is what we set out to show.

Similarly we may argue that
$$S_{13} = \rho\overline{uw} - g\overline{\int_0^{\xi} \rho(z-\zeta)\,d\xi'} \tag{5.6}$$

$$= \rho\overline{uw} - \rho N^2\overline{\xi\zeta}. \tag{5.7}$$

Now
$$k\xi + l\eta + m\zeta = 0 \tag{5.8}$$

from the continuity equation for a plane wave, and from the horizontal components of the equation of motion

$$\xi/k = \eta/l. \tag{5.9}$$

Thus, using (4.15),
$$S_{13} = \rho\overline{uw} + \frac{Ekm}{k^2+l^2} = \frac{Emc_1'}{\omega'} \tag{5.10}$$

and similarly
$$S_{23} = \rho\overline{vw} + \frac{Elm}{k^2+l^2} = \frac{Emc_2'}{\omega'}. \tag{5.11}$$

As already mentioned, T_{13} and T_{23} do not occur in the energy equation due to the restriction on the basic flow, and S_{13}, S_{23} have merely been calculated out of interest. It might be expected that the back effect of the waves on the mean flow could be considered in terms of the motion generated by a body force $-\partial(S_{ji})/\partial x_j$, but in fact the correct formulation is more complicated than this (see Bretherton 1969). It has also been pointed out to the author by R. W. Stewart (private communication) that S_{ij} cannot be regarded as a stress tensor in the usual sense, as it is not symmetric.

The second-order mean pressure associated with a plane wave cannot be calculated, but it would not enter the energy equation anyway as $\nabla . \mathbf{U} = 0$.

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